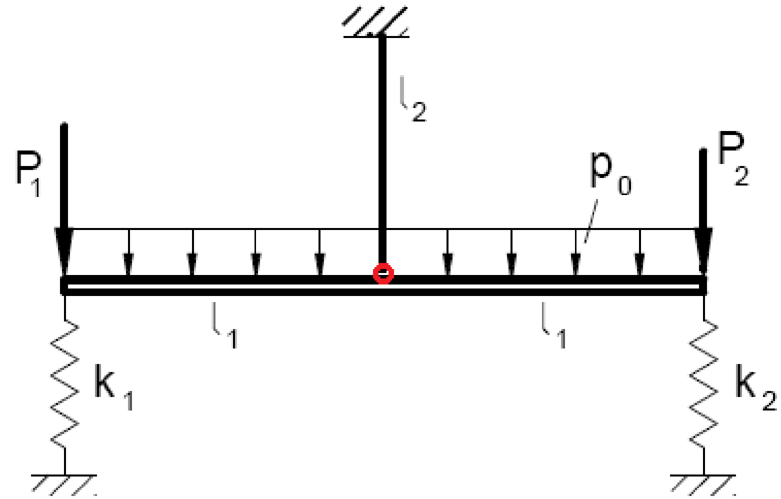
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

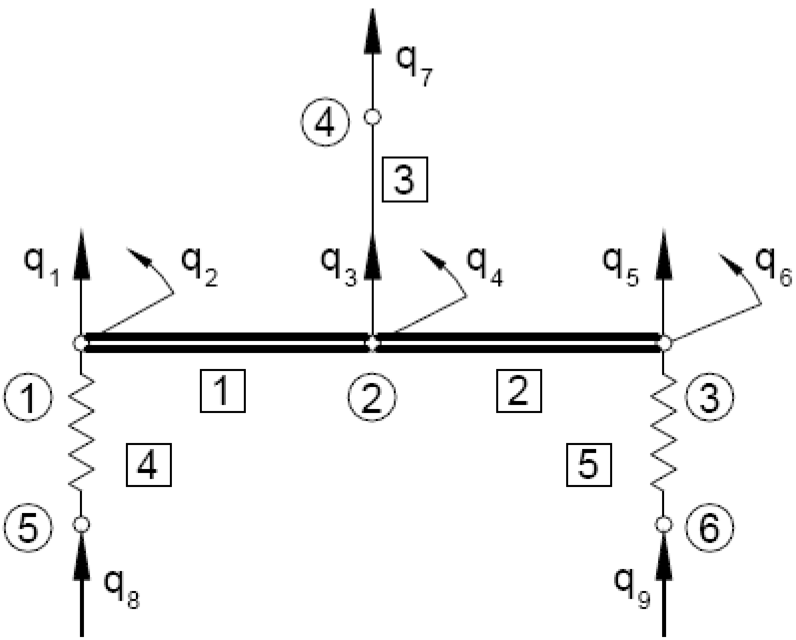
Finite element method (FEM1)

Lecture 10A. Beam element - examples

05.2025

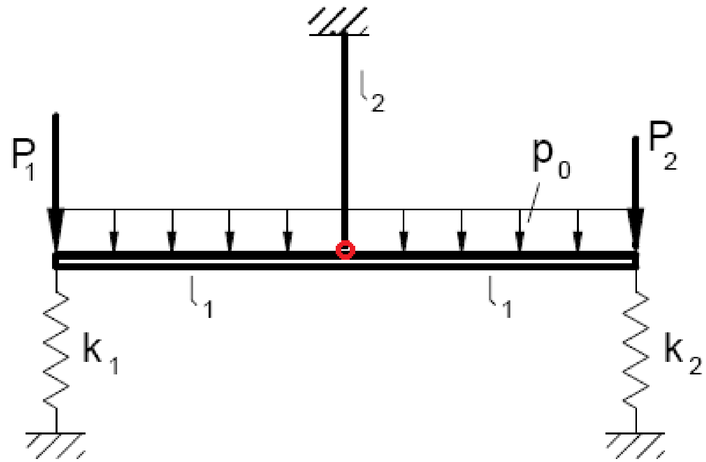
Example: A beam hanging on a rope and supported by two springs. Build a FEM model.



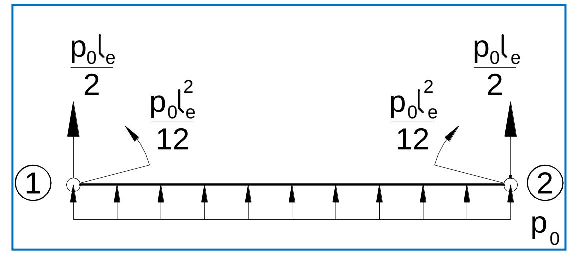


Global vector of nodal parameters:

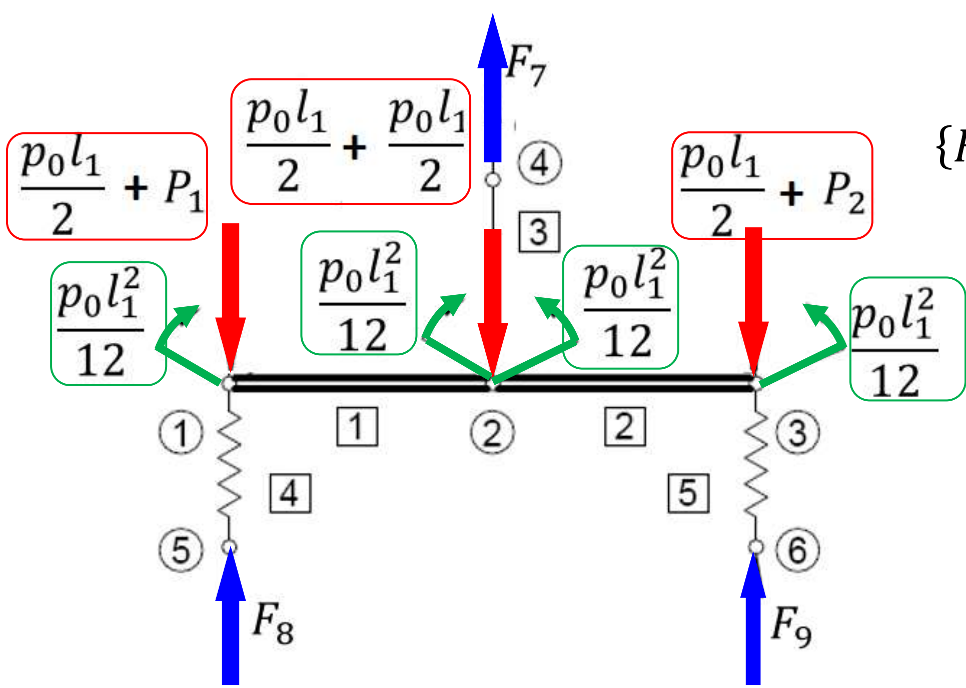
$$\{q\}_g = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_4 \\ q_4 \\ q_7 \\ q_8 \\ q_9 \end{Bmatrix}$$



Equivalent forces for constant transversal load:

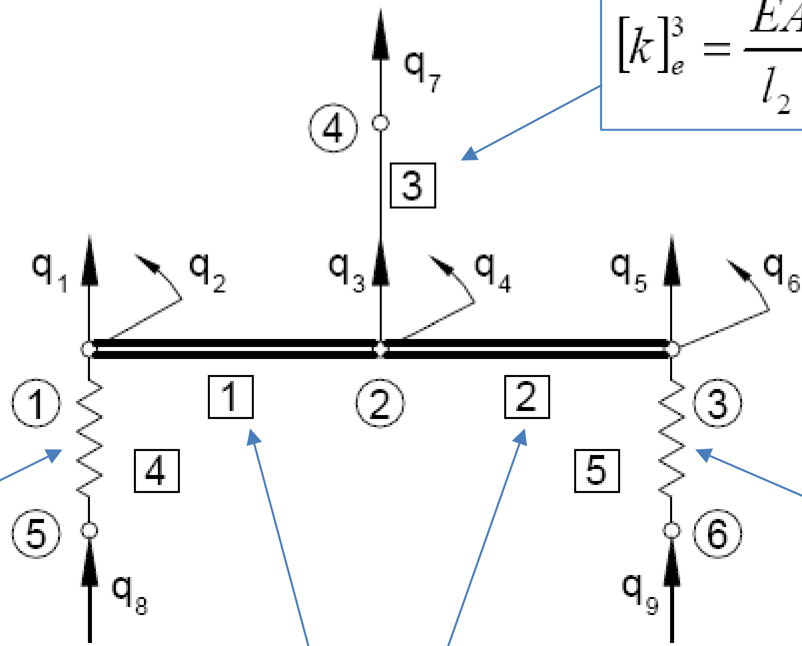


Global load vector:



$$\{F\}_g = \begin{Bmatrix} -\dfrac{p_0 l_1}{2} - P_1 \\ -\dfrac{p_0 l_1^2}{12} \\ -\dfrac{p_0 l_1}{2} - \dfrac{p_0 l_1}{2} \\ \dfrac{p_0 l_1^2}{12} - \dfrac{p_0 l_1^2}{12} \\ -\dfrac{p_0 l_1}{2} - P_2 \\ \dfrac{p_0 l_1^2}{12} \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{p_0 l_1}{2} - P_1 \\ -\dfrac{p_0 l_1^2}{12} \\ -p_0 l_1 \\ 0 \\ -\dfrac{p_0 l_1}{2} - P_2 \\ \dfrac{p_0 l_1^2}{12} \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix}$$

Element stiffness matrices:

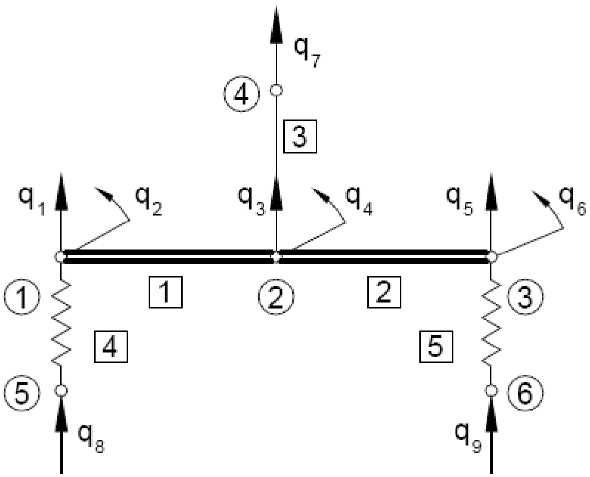


$$[k]_e^3 = \frac{EA}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k]_e^4 = k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k]_e^5 = k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k]_e^1 = [k]_e^2 = \frac{2EI}{l_1^3}\begin{bmatrix} 6 & 3l_1 & -6 & 3l_1 \\ 3l_1 & 2l_1^2 & -3l_1 & l_1^2 \\ -6 & -3l_1 & 6 & -3l_1^2 \\ 3l_1 & l_1^2 & -3l_1 & 2l_1^2 \end{bmatrix}$$



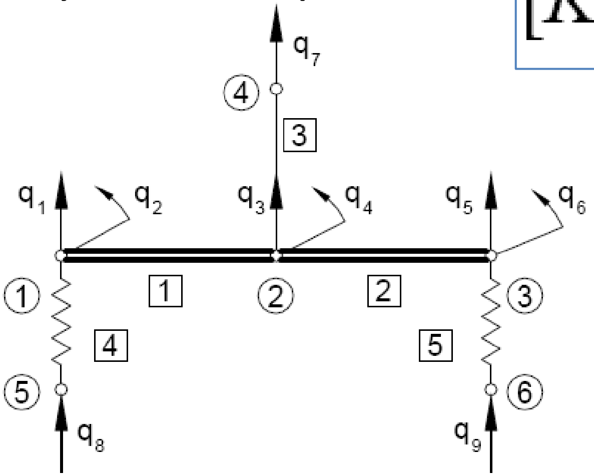
$$[k]_e^1 = [k]_e^2 = \frac{2EI}{l_1^3}\begin{bmatrix} 6 & 3l_1 & -6 & 3l_1 \\ 3l_1 & 2l_1^2 & -3l_1 & l_1^2 \\ -6 & -3l_1 & 6 & -3l_1^2 \\ 3l_1 & l_1^2 & -3l_1 & 2l_1^2 \end{bmatrix}$$

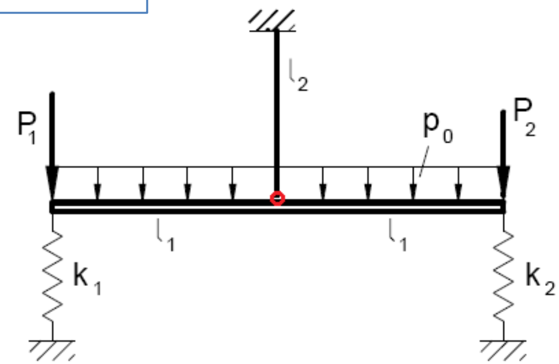
$$[k]_e^3 = \frac{EA}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^4 = k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^5 = k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

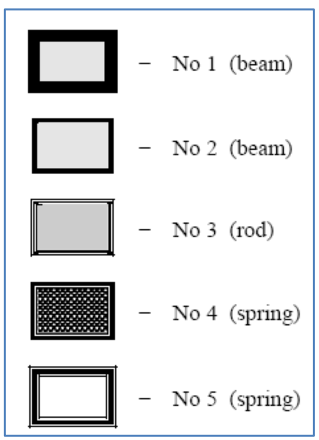
Global stiffness matrix:

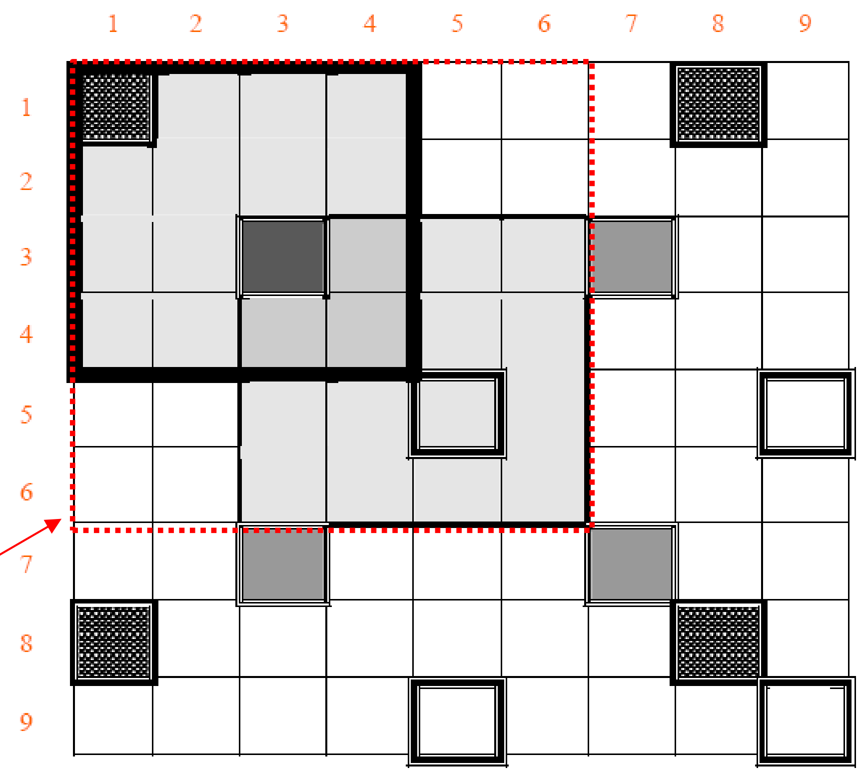
$$\underset{9\times 9}{[K]} = \begin{bmatrix} k_{11}^1 + k_{22}^4 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 & 0 & k_{12}^4 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 + k_{11}^3 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 & k_{12}^3 & 0 & 0 \\ k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{22}^5 & k_{34}^2 & 0 & 0 & k_{12}^5 \\ 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 & 0 & 0 & 0 \\ 0 & 0 & k_{21}^3 & 0 & 0 & 0 & k_{11}^3 & 0 & 0 \\ k_{21}^4 & 0 & 0 & 0 & 0 & 0 & 0 & k_{11}^4 & 0 \\ 0 & 0 & 0 & 0 & k_{21}^5 & 0 & 0 & 0 & k_{11}^5 \end{bmatrix}$$

System of equations: $[K]\{q\}=\{F\}$









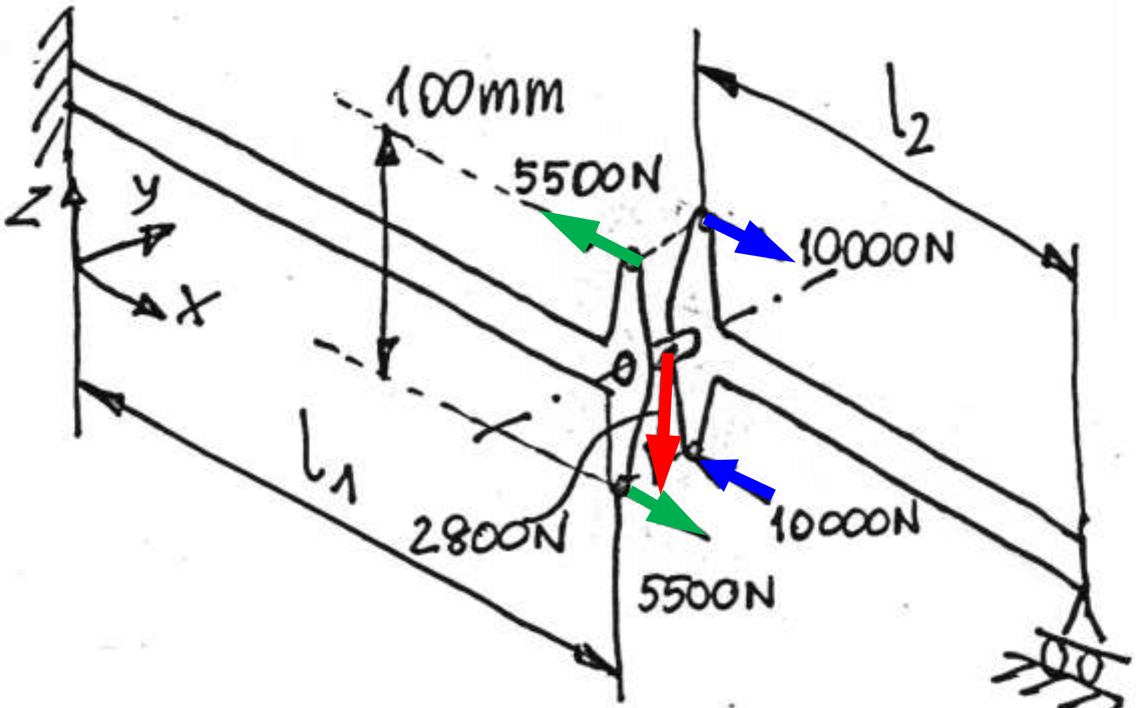
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-p_0 l_1}{2} - P_1 \\ \dfrac{-p_0 l_1^2}{12} \\ -p_0 l_1 \\ 0 \\ \dfrac{-p_0 l_1}{2} - P_2 \\ \dfrac{p_0 l_1^2}{12} \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix}$$

System of equations to be solved:

Example: Cantilever beam with a hinge.

Build a FEM model (use 2 elements).

Find reactions and internal forces. Check equilibrium conditions.



$E = 2 \cdot 10^5$ MPa

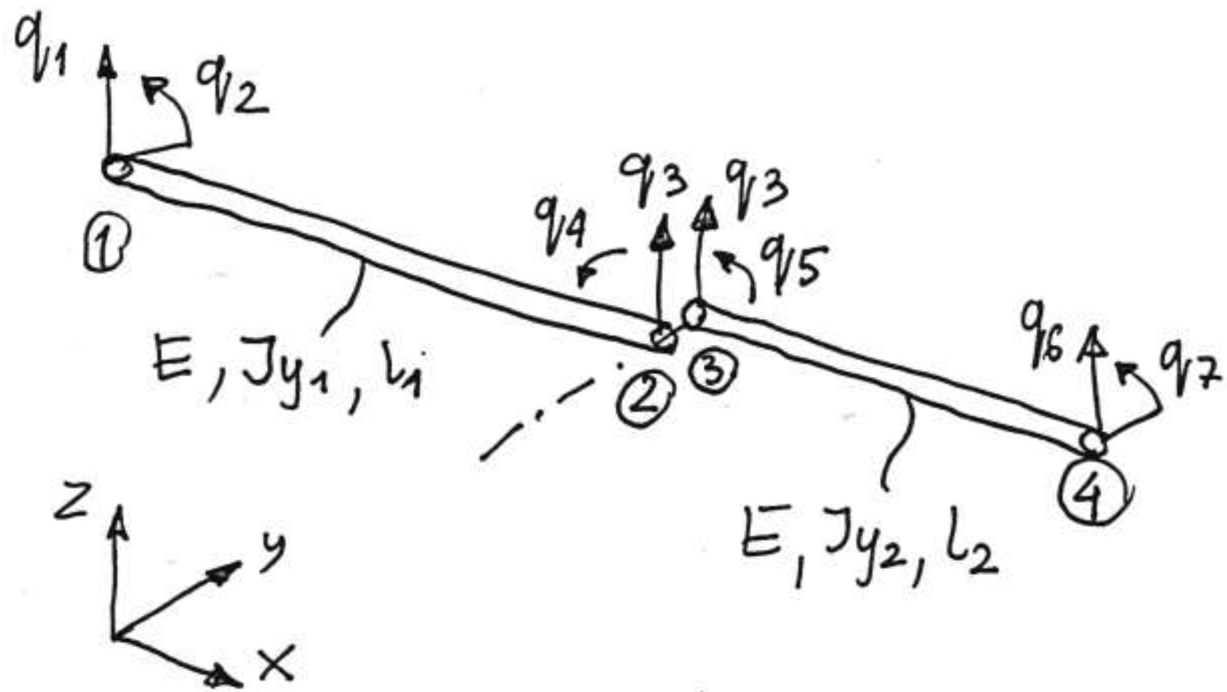
$l_1 = 1000$ mm

$l_2 = 500$ mm

$J_{y_1} = 1.143 \cdot 10^5$ mm^4

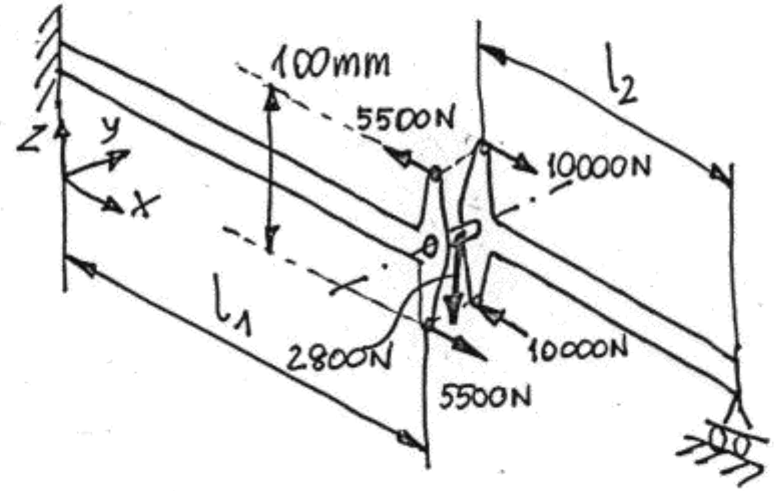
$J_{y_2} = 1.621 \cdot 10^5$ mm^4

Nodal parameters:

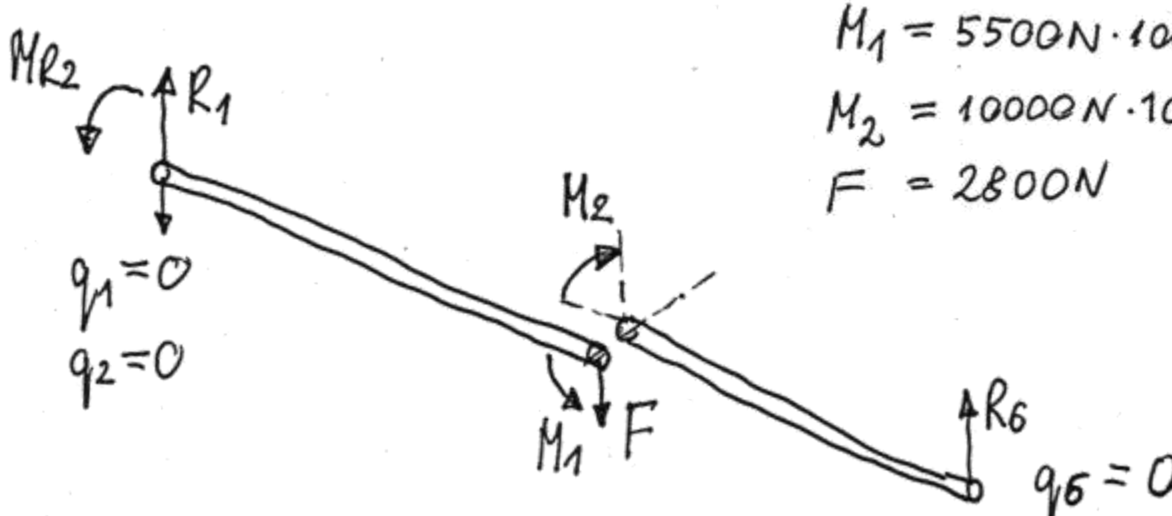


$$\underset{1\times 7}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7 \rfloor$$

Loads and reactions:



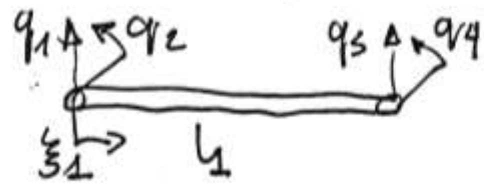
$$M_1 = 5500N \cdot 100mm = 0.55 \cdot 10^6\ Nmm$$
$$M_2 = 10000N \cdot 100\ mm = 1 \cdot 10^6\ Nmm$$
$$F = 2800N$$



$$\underset{1\times7}{\lfloor F \rfloor} = \lfloor R_1, M_{R2}, -F, M_1, -M_2, R_6, 0 \rfloor$$

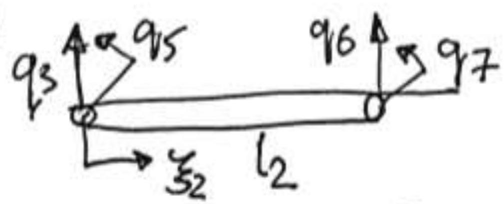
Stiffness matrices:

ELEMENT $\boxed{1}$

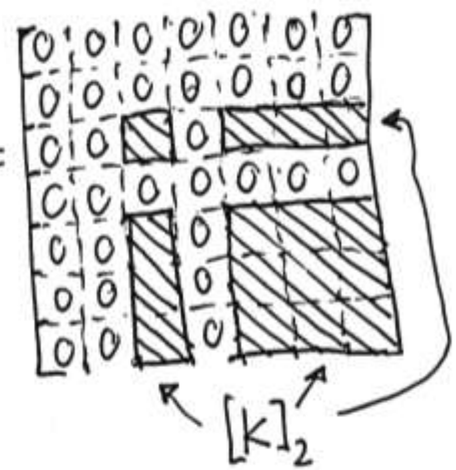


$$\underset{4\times 4}{[K]_1} = \frac{2EJ_{y1}}{l_1^3}\begin{bmatrix} 6 & 3l_1 & -6 & 3l_1 \\ 3l_1 & 2l_1^2 & -3l_1 & l_1^2 \\ -6 & -3l_1 & 6 & -3l_1 \\ 3l_1 & l_1^2 & -3l_1 & 2l_1^2 \end{bmatrix} ; \qquad \underset{7\times 7}{[K]_1^*} = \left[\begin{array}{c|c} [K]_1 & \underset{4\times 3}{[0]} \\ \hline \underset{3\times 4}{[0]} & \underset{3\times 3}{[0]} \end{array}\right]$$

ELEMENT $\boxed{2}$

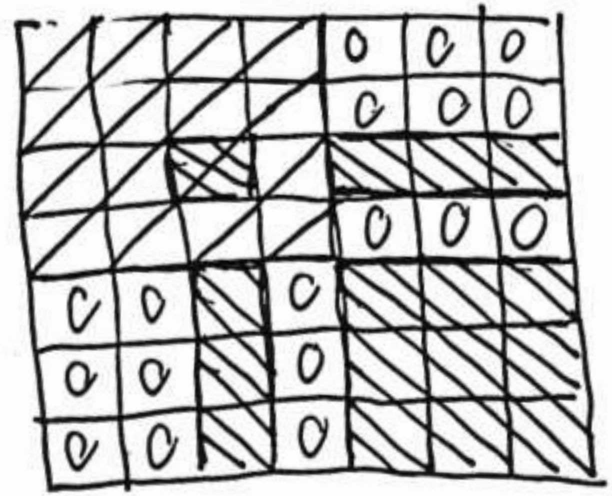


$$\underset{4\times 4}{[K]_2} = \frac{2EJ_{y2}}{l_2^3}\begin{bmatrix} 6 & 3l_2 & -6 & 3l_2 \\ 3l_2 & 2l_2^2 & -3l_2 & l_2^2 \\ -6 & -3l_2 & 6 & -3l_2 \\ 3l_2 & l_2^2 & -3l_2 & 2l_2^2 \end{bmatrix} ; \qquad \underset{7\times 7}{[K]_2^*} =$$



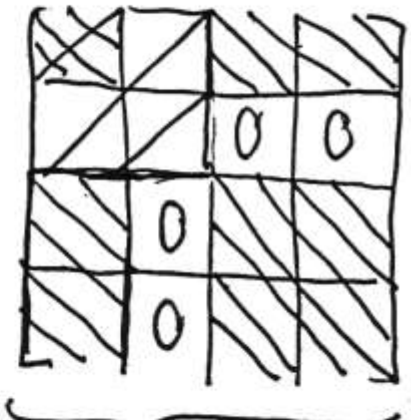
Global stiffness matrix:

$$\underset{7\times 7}{[K]} = [K]_1^* + [k]_2^* =$$



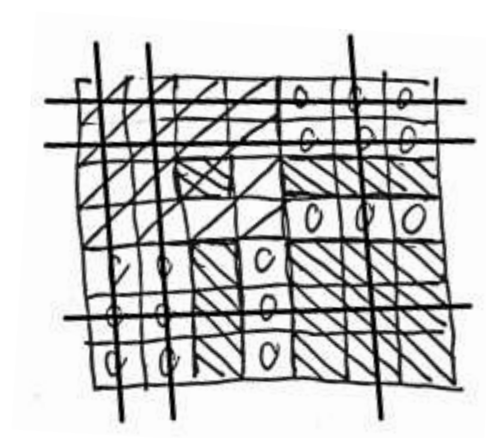
Boundary conditions:

$$q_1 = 0,\quad q_2 = 0,\quad q_6 = 0 \quad \Rightarrow \quad N = NDOF - NOF = 4$$



$$\cdot \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_7 \end{Bmatrix} = \begin{Bmatrix} -F \\ M_1 \\ -M_2 \\ 0 \end{Bmatrix}$$

$$[K]_{4\times 4}$$



$$
\underset{4\times4}{[K]} = \begin{bmatrix} \frac{12EJy_1}{l_1^3}+\frac{12EJy_2}{l_2^3} & -\frac{6EJy_1}{l_1^2} & \frac{6EJy_2}{l_2^2} & \frac{6EJy_2}{l_2^2} \\ -\frac{6EJy_1}{l_1^2} & \frac{4EJy_1}{l_1} & 0 & 0 \\ \frac{6EJy_2}{l_2^2} & 0 & \frac{4EJy_2}{l_2} & \frac{2EJy_2}{l_2} \\ \frac{6EJy_2}{l_2^2} & 0 & \frac{2EJy_2}{l_2} & \frac{4EJy_2}{l_2} \end{bmatrix}
$$

$$a = \frac{12EJy_1}{l_1^3} + \frac{12EJy_2}{l_2^3}$$

$$b = -\frac{6EJy_1}{l_1^2}, \quad c = \frac{6EJy_2}{l_2^2}$$

$$d = \frac{4EJy_1}{l_1}, \quad f = \frac{4EJy_2}{l_2}$$

$$
\underset{4\times4}{[K]} = \begin{bmatrix} a & b & c & c \\ b & d & 0 & 0 \\ c & 0 & f & f/2 \\ c & 0 & f/2 & f \end{bmatrix}
$$

$$\begin{cases} a q_3 + b q_4 + c q_5 + c q_7 = -F \\ b q_3 + d q_4 = M_1 \\ c q_3 + f q_5 + \frac{f}{2} q_7 = -M_2 \\ c q_3 + \frac{f}{2} q_5 + f \cdot q_7 = 0 \end{cases}$$

$\overline{\underline{\text{III}}} - \overline{\underline{\text{IV}}}$: $\frac{f}{2} q_5 - \frac{f}{2} q_7 = -M_2 \Rightarrow q_5 = q_7 - \frac{2M_2}{f}$

$\overline{\underline{\text{III}}}$: $c q_3 + f\left(q_7 - \frac{2M_2}{f}\right) + \frac{f}{2} q_7 = -M_2$

$c q_3 + \frac{3}{2} f \cdot q_7 = M_2 \Rightarrow q_7 = \frac{2(M_2 - c q_3)}{3f}$

$\overline{\underline{\text{II}}}$: $q_4 = \frac{M_1 - b q_3}{d}$

$q_5 = \frac{2(M_2 - c q_3)}{3f} - \frac{2M_2}{f} = -\frac{4M_2 + 2c q_3}{3f}$

I:

$$a \cdot q_3 + b\left(\frac{M_1 - b q_3}{d}\right) - c \cdot \frac{4M_2 + 2c q_3}{3f} + c \cdot \frac{2(M_2 - c q_3)}{3f} = -F$$

$$a \cdot q_3 + \frac{b}{d} M_1 - \frac{b^2}{d} q_3 - \frac{4M_2 c}{3f} - \frac{2c^2 q_3}{3f} + \frac{2M_2 c}{3f} - \frac{2c^2 q_3}{3f} = -F$$

$$\left(a - \frac{b^2}{d} - \frac{4c^2}{3f}\right) q_3 = -F - \frac{b}{d} M_1 + \frac{2cM_2}{3f}$$

$$q_3 = \frac{\frac{2cM_2}{3f} - F - \frac{b}{d} M_1}{\left(a - \frac{b^2}{d} - \frac{4c^2}{3f}\right)} = \frac{\frac{2 \cdot 6EJy_2 \cdot M_2 \cdot l_2}{3 \cdot l_2^2 \cdot 4EJy_2} - F + \frac{6EJy_1 \cdot M_1 l_1}{l_1^2 \cdot 4 \cdot EJy_1}}{\left(a - \frac{b^2}{d} - \frac{4c^2}{3f}\right)}$$

$$a - \frac{b^2}{d} - \frac{4c^2}{3f} = \frac{12EJy_1}{l_1^3} + \frac{12EJy_2}{l_2^3} - \frac{36E^2Jy_1^2}{l_1^4} \cdot \frac{l_1}{4EJy_1} - \frac{4 \cdot 36E^2Jy_2^2}{3 \cdot l_2^4} \cdot \frac{l_2}{4EJy_2} = \frac{3EJy_1}{l_1^3}$$

$$q_3 = \frac{\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) l_1^3}{3EJy_1}$$

Unknown nodal parameters:

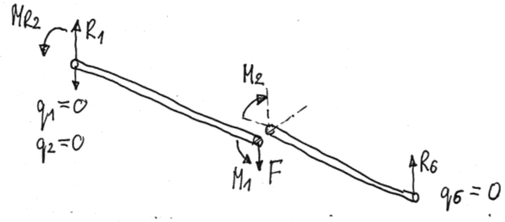
$$q_3 = \frac{\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) l_1^3}{3EJ_{y1}} = 0.3645\,mm$$

$$q_4 = \frac{\left(M_1 + \frac{M_2 l_1}{2 l_2} - \frac{F l_1}{2}\right) l_1}{EJ_{y1}} = 6.5617 \cdot 10^{-3} = 0.38^\circ$$

$$q_5 = -\frac{M_2 l_2 + \frac{J_{y2} l_1^3}{J_{y1} l_2}\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)}{3EJ_{y2}} = -5.8699 \cdot 10^{-3} = -0.34^\circ$$

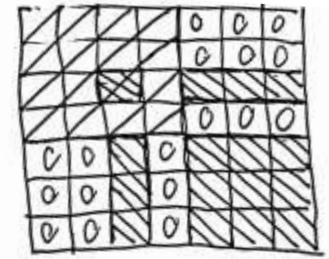
$$q_7 = \frac{\frac{1}{2} M_2 l_2 - \frac{J_{y2} l_1^3}{J_{y1} l_2}\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)}{3EJ_{y2}} = 1.8415 \cdot 10^{-3} = 0.11^\circ$$

Reactions:



$$\underset{1\times 7}{\lfloor F \rfloor} = \lfloor R_1, M_{R2}, -F, M_1, -M_2, R_6, 0 \rfloor$$

$$\underset{7\times 7}{[K]} =$$



$$R_1 = \frac{-12EJ_{y1}}{l_1^3}\cdot q_3 + \frac{6EJ_{y1}}{l_1^2}\cdot q_4 =$$

$$= -4\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) + \frac{6}{l_1}\left(M_1 + \frac{M_2 l_1}{2 l_2} - \frac{F l_1}{2}\right) = F - \frac{M_2}{l_2} = 800N$$

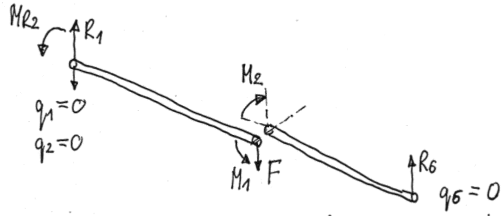
$$M_{R2} = -\frac{6EJ_{y1}}{l_1^2}\cdot q_3 + \frac{2EJ_{y1}}{l_1} q_4$$

$$= -2l_1\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) + 2\left(M_1 + \frac{M_2 l_1}{2 l_2} - \frac{F l_1}{2}\right) =$$

$$= -\left(3M_1 + 2\frac{M_2 l_1}{l_2} - 2Fl_1\right) + 2M_1 + M_2\frac{l_1}{l_2} - Fl_1 =$$

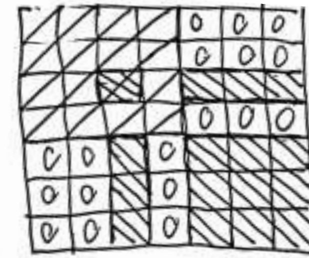
$$= Fl_1 - M_1 - M_2\frac{l_1}{l_2} = 0.25\cdot 10^6\, Nmm$$

Reactions
(continued):



$$\lfloor F \rfloor_{1\times 7} = \lfloor R_1, M_{R2}, -F, M_1, -M_2, R_6, 0 \rfloor$$

$$\underset{7\times7}{[K]} =$$



$$R_6 = \frac{-12EJ_{y2}}{l_2^3}\cdot q_3 + 0\cdot q_4 - \frac{6EJ_{y2}}{l_2^2}\cdot q_5 - \frac{6EJ_{y2}}{l_2^2}\cdot q_7 =$$

$$= -\frac{12EJ_{y2}}{l_2^3}\cdot\frac{\left(\frac{3}{2}\frac{M_1}{l_1}+\frac{M_2}{l_2}-F\right)l_1^3}{3EJ_{y1}} +$$

$$-\frac{6EJ_{y2}}{l_2^2}\cdot\left(-\frac{M_2 l_2 + \frac{J_{y2}}{J_{y1}}\frac{l_1^3}{l_2}\left(\frac{3}{2}\frac{M_1}{l_1}+\frac{M_2}{l_2}-F\right)}{3EJ_{y2}}\right) +$$

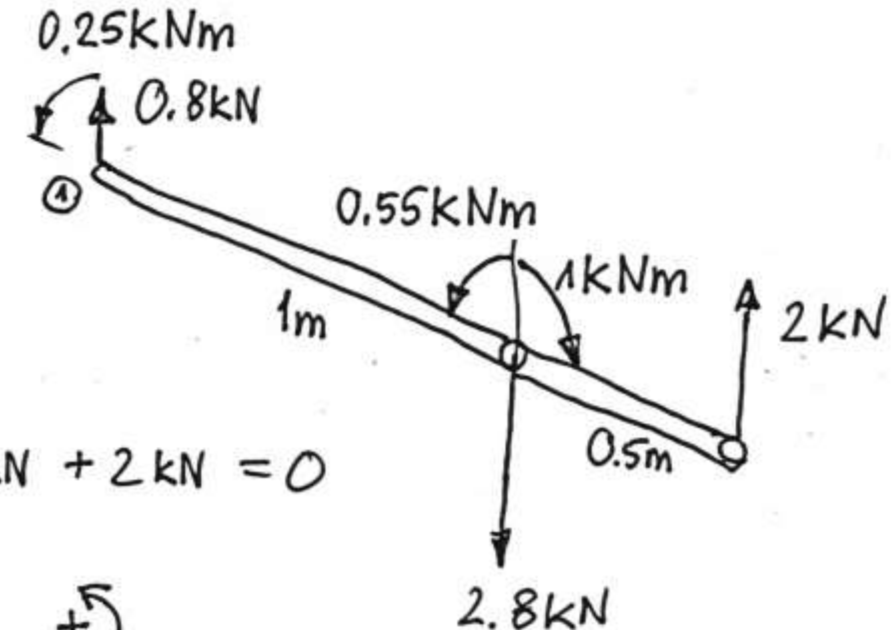
$$-\frac{6EJ_{y2}}{l_2^2}\cdot\left(\frac{\frac{1}{2}M_2 l_2 - \frac{J_{y2}}{J_{y1}}\frac{l_1^3}{l_2}\left(\frac{3}{2}\frac{M_1}{l_1}+\frac{M_2}{l_2}-F\right)}{3EJ_{y2}}\right) =$$

$$= -4\,\frac{J_{y2}}{J_{y1}}\frac{l_1^3}{l_2^3}\left(\frac{3}{2}\frac{M_1}{l_1}+\frac{M_2}{l_2}-F\right) +$$

$$+2\left(\frac{M_2}{l_2} + \frac{J_{y2}}{J_{y1}}\frac{l_1^3}{l_2^3}\left(\frac{3}{2}\frac{M_1}{l_1}+\frac{M_2}{l_2}-F\right)\right) +$$

$$-2\left(\frac{1}{2}\frac{M_2}{l_2} - \frac{J_{y2}}{J_{y1}}\frac{l_1^3}{l_2^3}\left(\frac{3}{2}\frac{M_1}{l_1}+\frac{M_2}{l_2}-F\right)\right) = \frac{M_2}{l_2} = 2\,000\ N$$

Equilibrium check:



$$\Sigma F_z = 0$$

$$0.8kN - 2.8kN + 2kN = 0$$

$$\Sigma M_y^{④} = 0 \quad +\circlearrowleft$$

$$0.25kNm + 0.55kNm - 1kNm - 2.8kN \cdot 1m + 2kN \cdot 1.5m = 0$$

Element solution:

Deflection in element 1:

$\boxed{1}$

$$w_1(\xi_1) = \underset{1\times 4}{\lfloor N \rfloor} \cdot \underset{4\times 1}{\{q\}_1} = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} =$$

$$= N_3 \cdot q_3 + N_4 \cdot q_4 =$$

$$= \left(\frac{3}{l_1^2} \cdot \xi_1^2 - \frac{2}{l_1^3} \cdot \xi_1^3 \right) \cdot q_3 + \left(\frac{1}{l_1^2} \cdot \xi_1^3 - \frac{1}{l_1} \xi_1^2 \right) q_4$$

$$w_1(0) = 0 \quad , \quad w_1(l_1) = (3-2) q_3 + (1-1) q_4 = q_3$$

$$\frac{dw_1}{d\xi} = \left(\frac{6}{l_1^2} \xi_1 - \frac{6}{l_1^3} \xi_1^2 \right) q_3 + \left(\frac{3}{l_1^2} \xi_1^2 - \frac{2}{l_1} \xi_1 \right) q_4$$

$$\left. \frac{dw_1}{d\xi_1} \right|_{625mm} = 0 \implies w_{1\,min} = w_1(625) = -0.712 mm$$

Bending moment in element 1:

$$M_{y_1}(\xi_1) = w_1'' E J_{y_1} = E J_{y_1} \cdot \left(N_3'' \cdot q_3 + N_4'' \cdot q_4 \right) =$$

$$= E J_{y_1} \cdot \left(\left(\frac{6}{l_1^2} - \frac{12}{l_1^3} \xi_1 \right) \cdot q_3 + \left(\frac{6}{l_1^2} \xi_1 - \frac{2}{l_1} \right) q_4 \right)$$

$$M_{y_1}(0) = -0.25 \cdot 10^6 \, Nmm$$

$$M_{y_1}(l_1) = 0.55 \cdot 10^6 \, Nmm = M_1$$

Shear force in element 1:

$$T_{z_1}(\xi_1) = -w_1''' E J_{y_1} = -E J_{y_1} \left(N_3''' \cdot q_3 + N_4''' \cdot q_4 \right) =$$

$$= +E J_{y_1} \cdot \left(\frac{12}{l_1^3} q_3 - \frac{6}{l_1^2} q_4 \right) = -800 N$$

$\boxed{2}$

Deflection in element 2:

$$w_2(\xi_2) = \lfloor N \rfloor \cdot \{q\}_2 = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} q_3 \\ q_5 \\ 0 \\ q_7 \end{Bmatrix} =$$

$$= N_1 \cdot q_3 + N_2 \cdot q_5 + N_4 \cdot q_7 =$$

$$= \left(1 - \frac{3}{l_2^2}\xi_2^2 + \frac{2}{l_2^3}\cdot\xi_2^3\right) q_3 + \left(\xi_2 - \frac{2}{l_2}\xi_2^2 + \frac{1}{l_2^2}\xi_2^3\right) q_5 + \left(\frac{1}{l_2^2}\xi_2^3 - \frac{1}{l_2}\xi_2^2\right) q_7$$

$$w_2(0) = q_3$$

$$w_2(l_2) = (1 - 3 + 2) q_3 + (l_2 - 2l_2 + l_2) q_5 + (l_2 - l_2) q_7 = 0$$

$$\frac{dw_2}{d\xi_2} = \left(\frac{6}{l_2^3}\xi_2^2 - \frac{6}{l_2^2}\xi_2\right) q_3 + \left(1 - \frac{4}{l_2}\xi_2 + \frac{3}{l_2^2}\xi_2^2\right) q_5 + \left(\frac{3}{l_2^2}\xi_2^2 - \frac{2}{l_2}\xi_2\right) q_7$$

$$\left.\frac{dw_2}{d\xi_2}\right|_{255.7\,mm} = 0 \quad \Rightarrow w_{2\,min} = w_2(255.7) = -0.3\,mm$$

Bending moment in element 2:

$$M_{y2}(\xi_2) = w_2'' EJ_{y2} = EJ_{y2}\left(N_1''\cdot q_3 + N_2''\cdot q_5 + N_4''\cdot q_7\right) =$$

$$= EJ_{y2}\cdot\left(\left(\frac{12}{l_2^3}\xi_2 - \frac{6}{l_2^2}\right)q_3 + \left(\frac{6}{l_2^2}\xi_2 - \frac{4}{l_2}\right)q_5 + \left(\frac{6}{l_2^2}\xi_2 - \frac{2}{l_2}\right)q_7\right)$$

$$M_{y2}(0) = 1\cdot 10^6\ Nmm = M_2$$

$$M_{y2}(l_2) = 0$$

Shear force in element 2:

$$T_{z2} = -w_2''' \cdot EJ_{y2} = -EJ_{y2}\cdot\left(N_1'''\cdot q_3 + N_2'''\cdot q_5 + N_4'''\cdot q_7\right) =$$

$$= -EJ_{y2}\cdot\left(\frac{12}{l_2^3}q_3 + \frac{6}{l_2^2}q_5 + \frac{6}{l_2^2}q_7\right) = 2000\ N$$

Deformation and internal forces: